Señales y Sistemas

Tema II:
Respuesta de Sistemas LTI en el dominio del tiempo

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La señal respuesta al impulso h[n]:

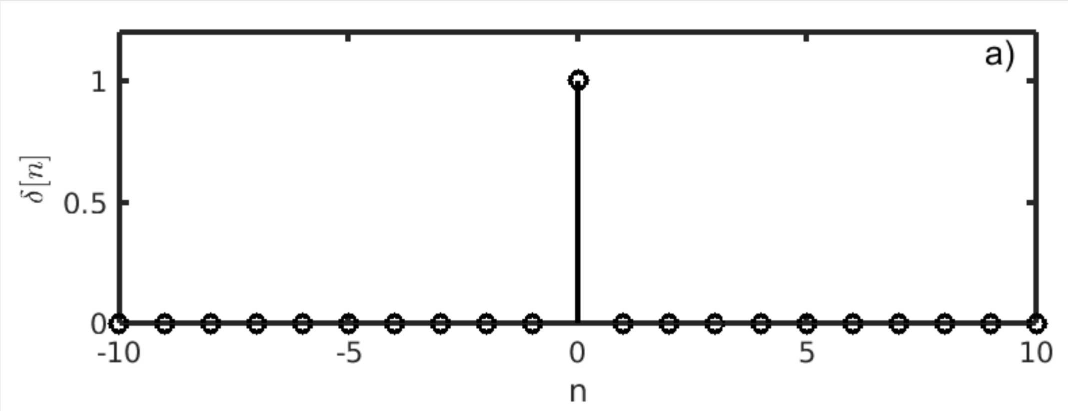
Señal Respuesta al Impulso → $h[n] \equiv T\{\delta[n]\}$

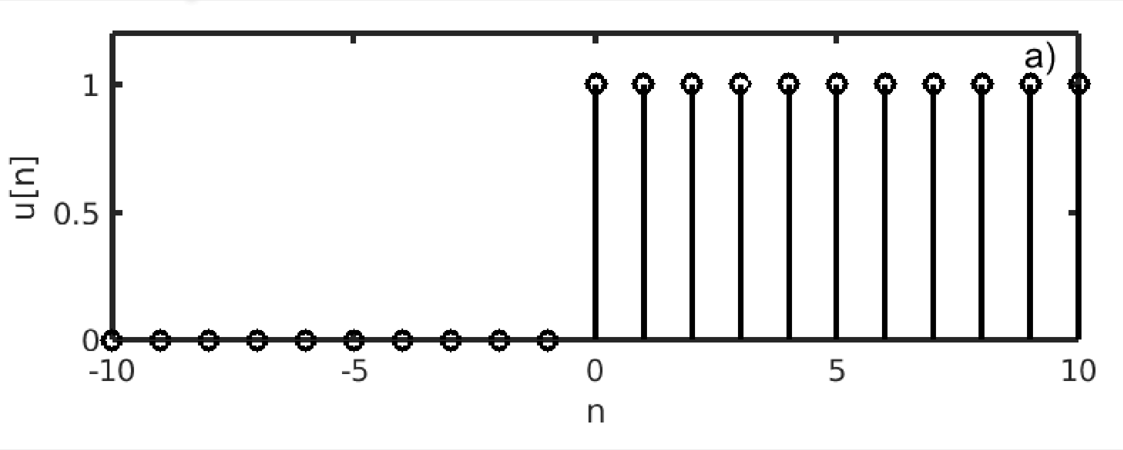
Ejemplo: respuesta al impulso del acumulador

$$y[n] = \sum_{k=-\infty}^{n} x[k].$$

$$h[n] = T\{\delta[n]\} = \sum_{k=-\infty}^{n} \delta[k] = u[n]$$

$$h[n] = u[n]$$



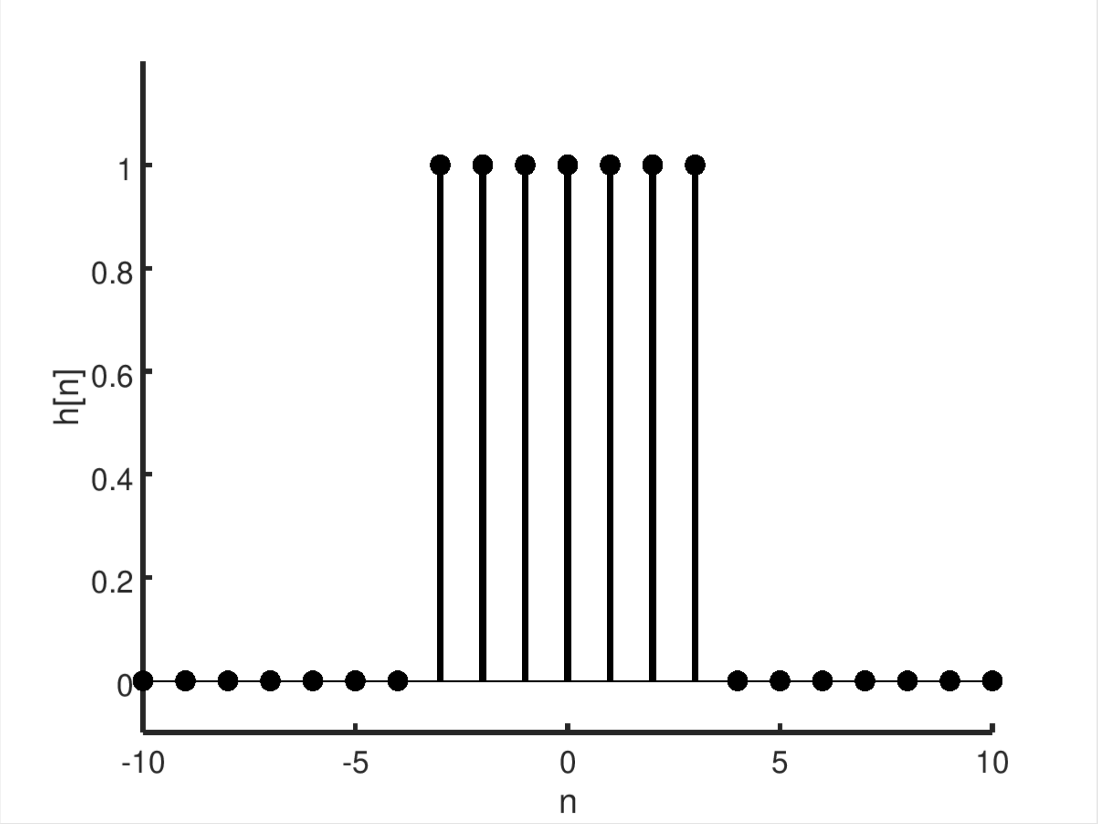


Ejemplo: respuesta al impulso del promediador móvil

$$y[n] = \frac{1}{7} \sum_{k=-3}^{k=3} x[n-k].$$

$$h[n] = T\{\delta[n]\} = \frac{1}{7} \sum_{k=-3}^{k=3} \delta[n-k].$$

$$h[n] = \frac{1}{7} \left\{1, 1, 1, \underline{1}, 1, 1, 1\right\}.$$



Respuesta de sistemas LTI discretos:

Propiedad de selección → $$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$y[n] = T\{x[n]\} = T\left\{ \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \right\},$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\},$$

$$h[n-k] = T\{\delta[n-k]\}.$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

Suma de convolución → $$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Suma de convolución → $$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

La salida de un sistema LTI discreto está completamente caracterizada por señal Respuesta al Impulso h[n], en otras palabras, **si conocemos h[n] podremos predecir la respuesta del sistema ante cualquier entrada x[n]**.

Ejemplo: sistema retardo

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$h[n] = \delta[n - n_0]$$

$$y[n] = x[n] * \delta[n - n_0] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - n_0 - k]$$

$$= x[n - n_0]$$

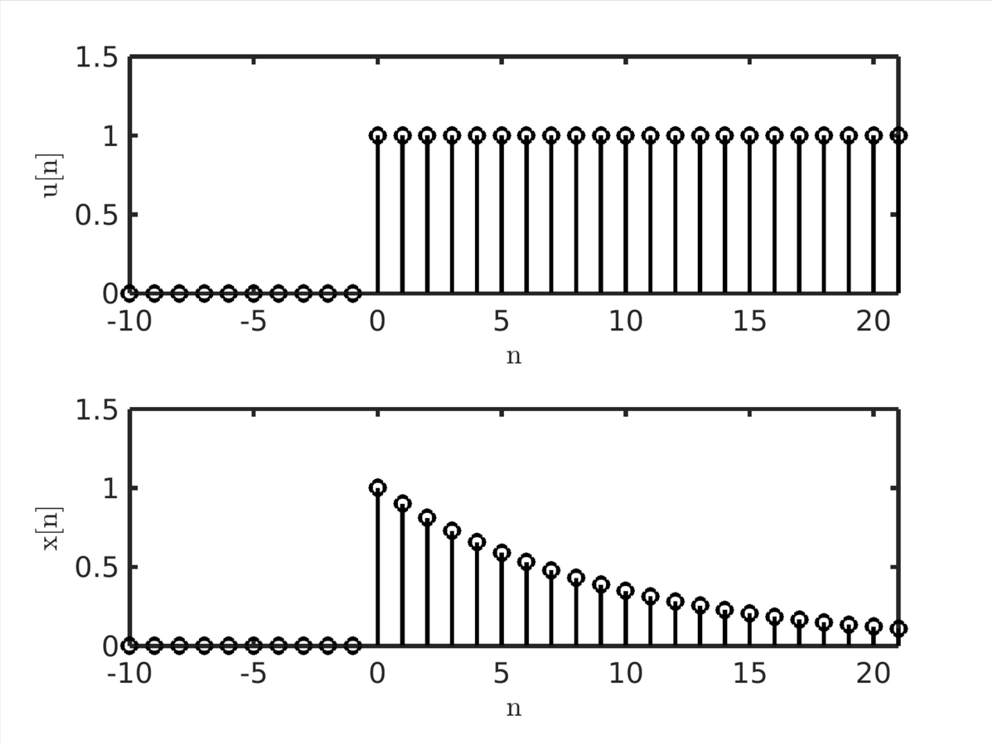
$$x[n] * \delta[n - n_0] = x[n - n_0]$$

Ejemplo: sistema acumulador

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$h[n] = u[n].$$

$$x[n] = \alpha^n u[n] \quad \text{con} \quad 0 < \alpha < 1$$



$$y[n] = \sum_{k=-\infty}^{\infty} \alpha^k u[k]u[n-k]$$

$$u[k]u[n-k] = 1 \begin{cases} k \geq 0 \\ n-k \geq 0 \quad k < n \end{cases}$$

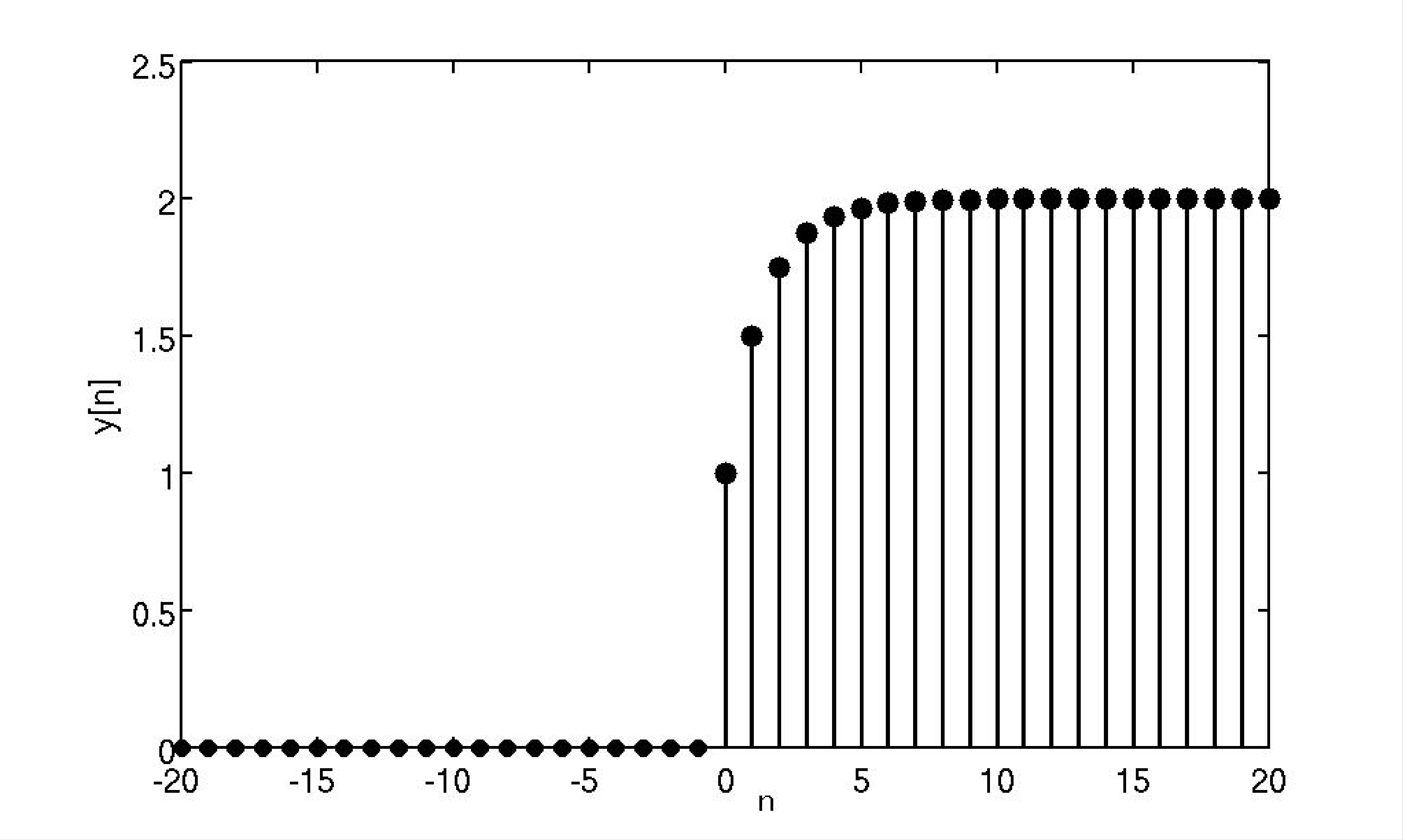
$$y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha} \quad n \geq 0$$

$$y[n] = 0, \quad n < 0$$

$$y[n] = \frac{1-\alpha^{n+1}}{1-\alpha} u[n].$$

$h[n] = u[n].$ $\qquad x[n] = \alpha^n u[n] \;\; \text{con} \;\; 0 < \alpha < 1$

$$y[n] = x[n] * u[n] = \frac{1 - \alpha^{(n+1)}}{1 - \alpha} u[n]$$



La señal respuesta al impulso h(t) para sistemas continuos:

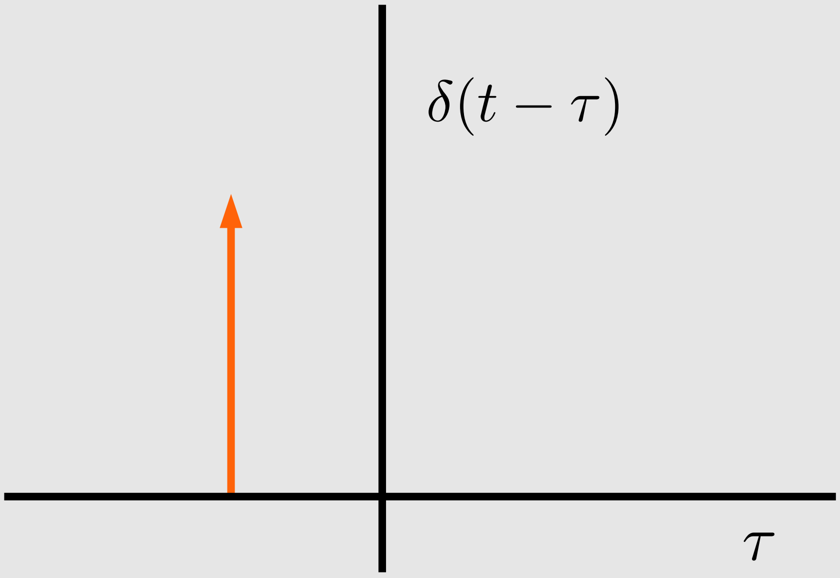
Señal Respuesta al Impulso → $h(t) = T\left\{\delta(t)\right\}$

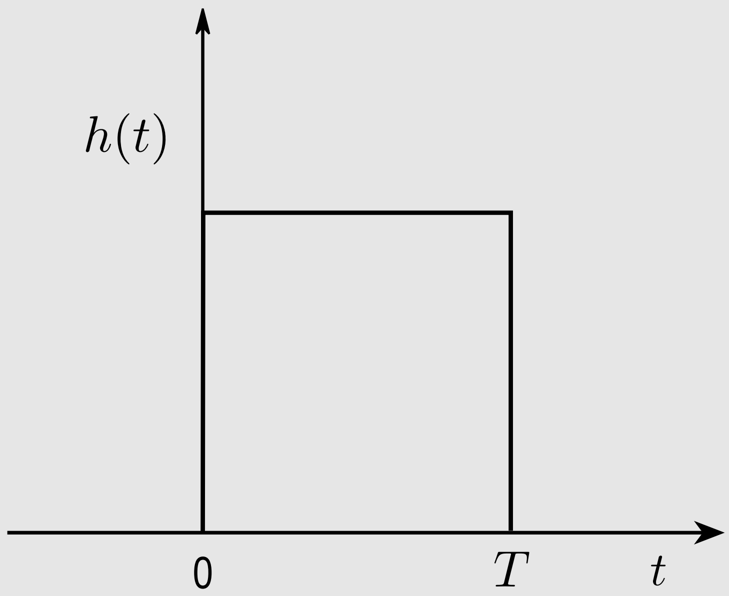
Ejemplo: promediador móvil continuo

$$y(t) = \frac{1}{T} \int_0^T x(t-\tau)d\tau.$$

$$h(t) = \mathcal{T}\{\delta(t)\} = \frac{1}{T} \int_0^T \delta(t-\tau)d\tau.$$

$$h(t) = u(t) - u(t-T)$$





Ejemplo: sistema LTI sobre circuito RC

$$\frac{dV_c(t)}{dt} + \frac{1}{RC}\, V_c(t) = \frac{1}{RC}\, V_s(t) \qquad \text{Cond. inicial } V_c(0) = 0$$

$$V_c(t) = \frac{1}{RC} \left(\int_0^t V_s(\tau)\, e^{\tau/RC}\, d\tau \right) e^{-t/RC}$$

$$h(t) = \left(\int_0^t \delta(\tau)\, e^{\tau/RC}\, d\tau \right) e^{-t/RC}$$

$$h(t) = \frac{1}{RC}\, e^{-t/RC}$$

Respuesta de sistemas LTI continuos:

Propiedad de selección → $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$

$$y(t) = T\{x(t)\} = T\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\}.$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)T\left\{\delta(t-\tau)\right\}d\tau, \qquad h(t-\tau) = T\{\delta(t-\tau)\}.$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

Integral de convolución → $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

Integral de convolución

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

La salida de un sistema LTI continuo está completamente caracterizada por señal Respuesta al Impulso h(t), en otras palabras, **si conocemos h(t) podremos predecir la respuesta del sistema ante cualquier entrada x(t).**

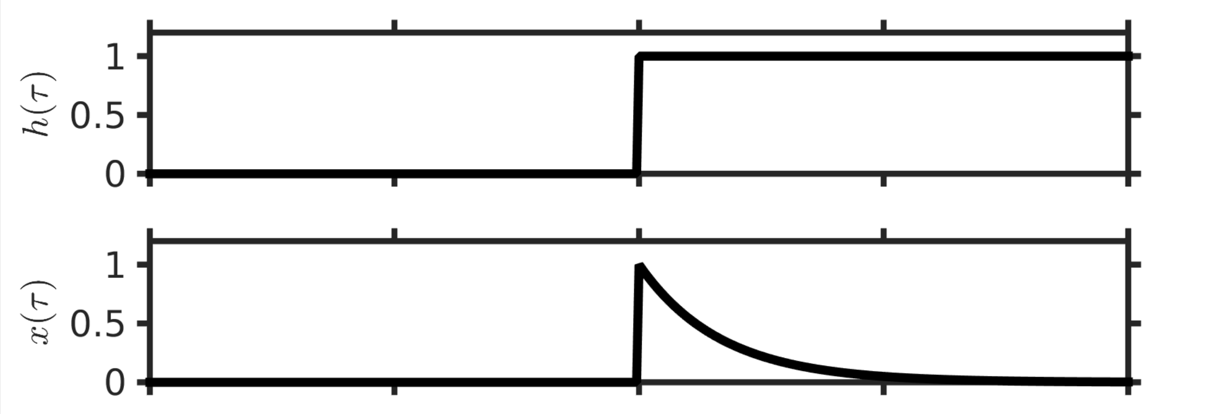
Convolución de señales

$$x_3(t) = x_2(t) * x_1(t) = \int_{-\infty}^{\infty} x_2(\tau)x_1(t-\tau)d\tau,$$

$$x_3[n] = x_2[n] * x_1[n] = \sum_{k=-\infty}^{\infty} x_2[k]x_1[n-k]$$

Ejemplo

$$x(t) = e^{-at}u(t) \quad a > 0$$
$$h(t) = u(t)$$

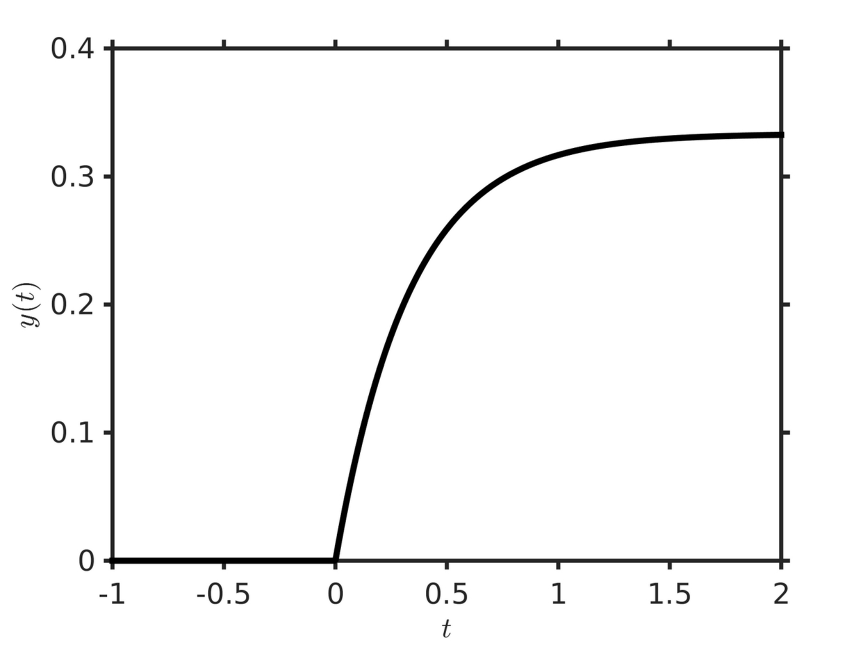


$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{\infty} e^{-a\tau} u(\tau)\, u(t-\tau)\, d\tau$$

$$u(\tau)u(t-\tau) = 1 \begin{cases} \tau > 0 \\ t - \tau > 0 \;\Rightarrow\; \tau < t \end{cases}$$

$$y(t) = \left\{ \int_0^t e^{-a\tau} d\tau \right\} u(t)$$



$$y(t) = \frac{1 - e^{-at}}{a} u(t)$$

Propiedades de la operación convolución:

Operación convolución →

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau$$

$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$

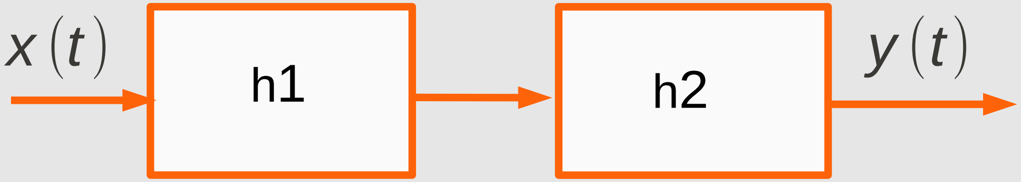
conmutativa → $x_1(t) * x_2(t) = x_2(t) * x_1(t)$

asociativa → $x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$

distributiva → $x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t)$

asociativa

$$y(t) = [x(t) * h_1(t)] * h_2(t)$$

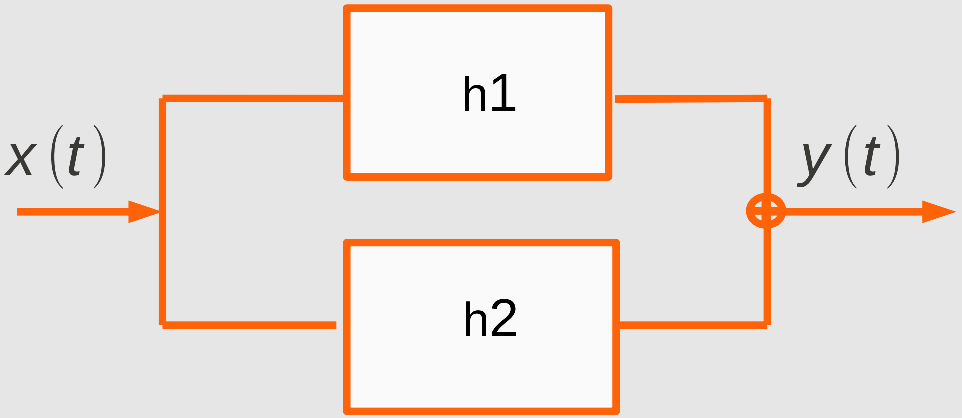


$$y(t) = x(t) * [h_1(t) * h_2(t)]$$

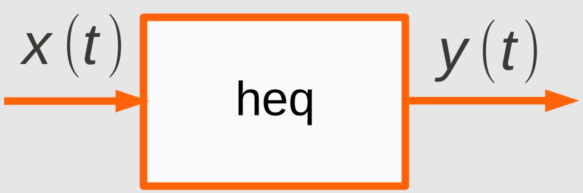
$$h_{eq}(t) \equiv h_1(t) * h_2(t)$$

distributiva

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$



$$y(t) = x(t) * [h_1(t) + h_2(t)]$$



$$h_{eq}(t) \equiv h_1(t) + h_2(t)$$

Relación entre la Resp. al Impulso y las características de un LTI:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Sistemas con y sin memoria:

Un sistema **no posee memoria** si la salida en todo instante depende sólo de los valores de la entrada en ese mismo instante.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$\Rightarrow \quad h[n-k] = 0, \quad \text{si } k \neq n$$

$$\Rightarrow \quad h[n] = \alpha\delta[n]$$

Sistemas sin memoria: $\Rightarrow \quad h[n] = \alpha\delta[n] \quad h(t) = \alpha\delta(t)$

$$h[n] = \frac{1}{3}\left(\delta[n-1] + \delta[n] + \delta[n+1]\right)$$

$$h[n] = u[n], \qquad h(t) = u(t) - u(t-1), \qquad h[n] = 3\delta[n]$$

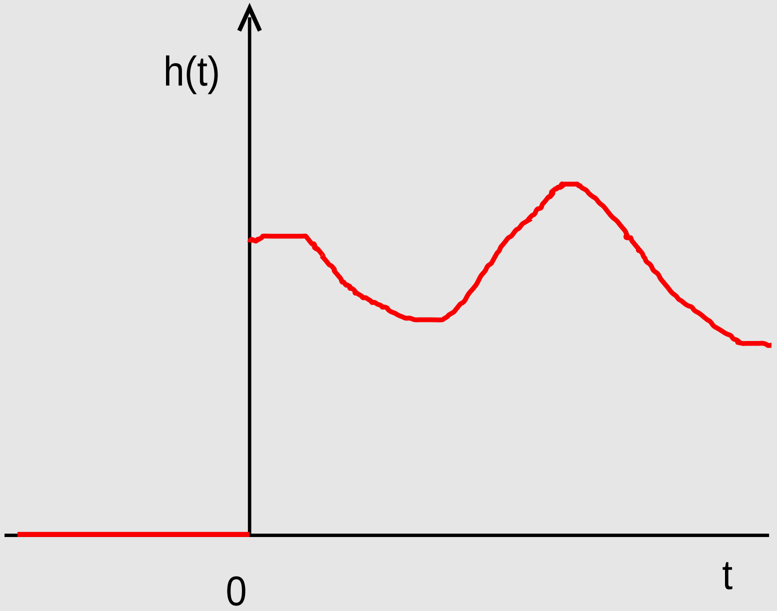
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k], \qquad y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Sistemas causales y no causales:

Un sistema es causal si la salida a tiempo t_0 depende sólo de la entrada a valores de $t \leq t_0$.

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

$$\Rightarrow \quad h(t-\tau) = 0 \quad \tau > t, \quad h(\zeta) = 0,\ \zeta < 0.$$



Sistemas causales: $\Rightarrow \quad h[n] = 0,\ n < 0 \quad h(t) = 0,\ t < 0$

Sistemas causales: $\Rightarrow \quad h[n] = 0,\ n < 0 \qquad h(t) = 0,\ t < 0$

$$¿h[n] = u[n]? \qquad ¿h(t) = \frac{1}{T}\left(u(t+T/2) - u(t-T/2)\right)?$$

Definición:

Condición de reposo inicial (*initial rest*):
$$\begin{cases} x[n] = 0,\ n < n_0 \quad \Rightarrow \quad y[n] = 0,\ n < n_0 \\ x(t) = 0,\ t < t_0 \quad \Rightarrow \quad y(t) = 0,\ t < t_0 \end{cases}$$

LTI causal $\Leftrightarrow$ LTI en reposo inicial

Respuesta al impulso de sistemas LTI estables.

$$\text{para toda señal que cumpla} \;\; |x[n]| < C \;\; \forall n \quad \rightarrow \quad |y[n]| < D \;\; \forall n$$

$$\begin{aligned} |y[n]| &= \left|\sum_{k=-\infty}^{\infty} x[k]h[n-k]\right| \\ &\leq \sum_{k=-\infty}^{\infty} |x[k]||h[n-k]| \\ &< C \sum_{k=-\infty}^{\infty} |h[n-k]| \end{aligned}$$

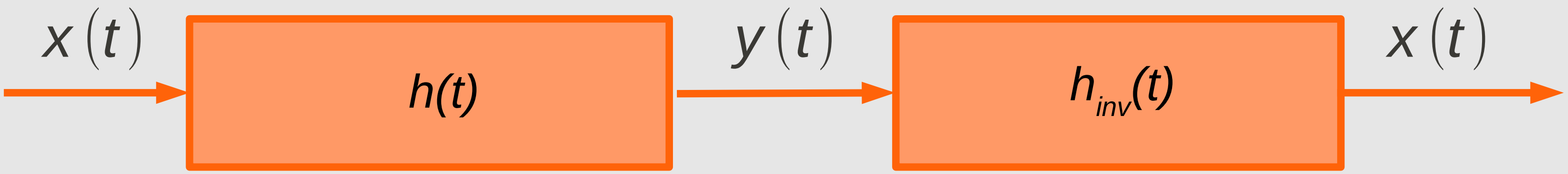
Entonces para un sistema LTI estable debe converger la suma

$$\sum_{k=-\infty}^{\infty} |h[n]| \qquad\qquad \int_{-\infty}^{\infty} |h(t)|\, dt$$

$$\text{sistema LTI estable} \iff \sum_{k=-\infty}^{\infty} |h[n]| < \infty \quad \int_{-\infty}^{\infty} |h(t)| dt < \infty$$

$$¿h[n] = u[n]? \qquad ¿h(t) = \frac{1}{T}\left(u(t+T/2) - u(t-T/2)\right)?$$

Sistemas inverso:



$$x(t) = T_{inv}\left\{T\left\{x(t)\right\}\right\}$$

$$h_{eq}(t) \equiv h(t) * h_{inv}(t)$$

$$h[n] * h_{inv}[n] = \delta[n], \qquad h(t) * h_{inv}(t) = \delta(t)$$

Respuesta al escalón:

$$s(t) \equiv T\{u(t)\} \qquad \text{o} \qquad s[n] \equiv T\{u[n]\}$$

Sistema LTI discreto →

$$s[n] = \sum_{k=-\infty}^{n} h[k] \qquad s[n] - s[n-1] = h[n]$$

Sistema LTI continuo →

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau \qquad h(t) = \frac{ds(t)}{dt}$$

Sistemas LTI continuos causales descripto por ecuaciones diferenciales:

$$\frac{d^N y(t)}{dt^N} + \sum_{k=0}^{N-1} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Se necesita especificar las C.I → $y(t_0), y'(t_0), ..., y^{N-1}(t_0)$

$$\frac{dy(t)}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} x(t), \qquad y(t_0) = y_0$$

Ecuación diferencial lineal, con coeficientes constantes y que cumple:

$$y(t_0), y'(t_0), ..., y^{N-1}(t_0) = 0$$

$$x(t) = 0, \quad t < t_0 \quad \Rightarrow \quad y(t) = 0, \quad t < t_0 \quad \text{reposo inicial}$$

Sistema LTI causal

Respuesta al Impulso de un sistema de orden N

$$\frac{d^N y(t)}{dt^N} + \sum_{k=0}^{N-1} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$\frac{d^N h(t)}{dt^N} + \sum_{k=0}^{N-1} a_k \frac{d^k h(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k \delta(t)}{dt^k}$$

La respuesta al impulso de un sistema es la respuesta del sistema cuando la entrada es un impulso, con las condiciones iniciales nulas.

$$h(t) = T\left\{\delta(t)\right\} \quad + \quad c.i \;\; \text{nulas}$$

Sistemas LTI discretos causales descriptos por ecuaciones en diferencias:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Se necesita especificar las C.I. → $y[n_0-1], y[n_0-2], \ldots, y[n_0-N]$

$$y[n] - y[n-1] = x[n] \qquad y[n_0-1] = y_0, \quad x[n] = 0, \quad n < 0$$

Sistema LTI causal ⇐

Sistemas LTI: ecuación en diferencias lineal y con coeficientes constantes

$$x[n] = 0 \;\; (n < n_0) \quad \Rightarrow \quad y[n] = 0 \;\; (n < n_0)$$

$$y[n_0-1], y[n_0-2], \ldots, y[n_0-N] = 0$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k], \qquad x[n] = 0, \qquad n < 0$$

$$y[n] = y_h[n] + y_p[n]$$

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\}$$

C.I.

$$y[0] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[-k] \overbrace{+ a_1 y[-1] + a_2 y[-2] + \ldots + a_N y[-N]}\right\}$$

$$y[1] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[1-k] + a_1 y[0] + a_2 y[-1] + \ldots + a_N y[1-N]\right\}$$

....

$$y[n] = \frac{1}{a_0} \left\{ \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k] \right\}$$

N=0,
sistema no-recursivo ⟹

$$y[n] = \frac{1}{a_0} \sum_{k=0}^{M} b_k x[n-k]$$

N≠0,
sistema recursivo ⟹

$$y[n] = \frac{1}{a_0} \left\{ \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k] \right\}$$

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\}$$

$$h[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k \delta[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\}$$

N=0 ⇒ Sistemas FIR

$$h[n] = \begin{cases} b_n/a_0 & 0 \leq n \leq M \\ 0 & \text{en cualquier otro caso} \end{cases}$$

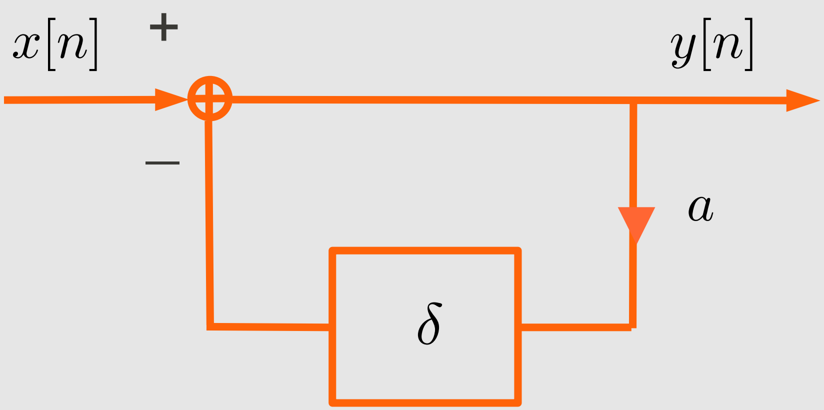
N≠0 ⇒ Sistemas IIR

$h[n] \rightarrow$ número infinito de términos distintos de 0

Representación de sistemas mediante diagramas de bloques:

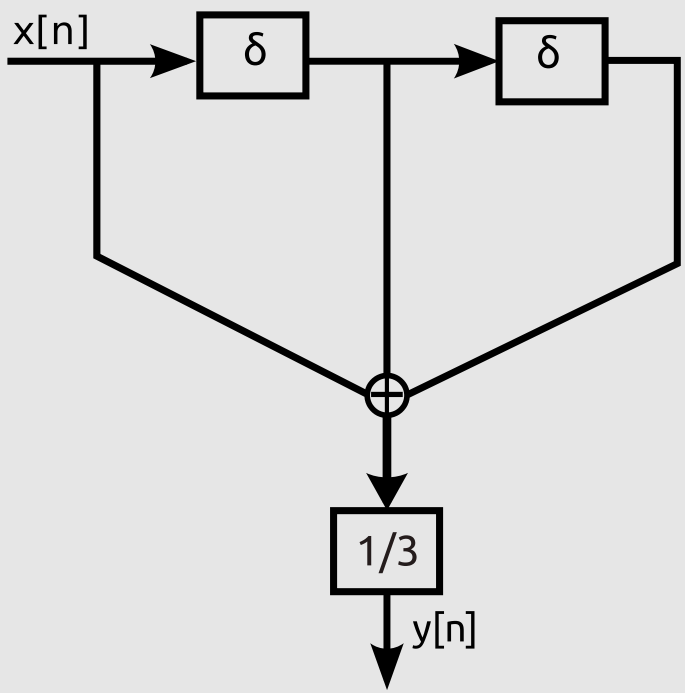
Sistemas discretos:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$



$$y[n] + ay[n-1] = x[n]$$

$$y[n] = x[n] - ay[n-1]$$

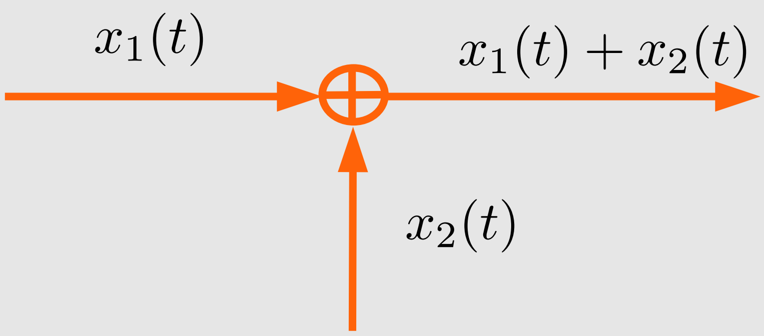


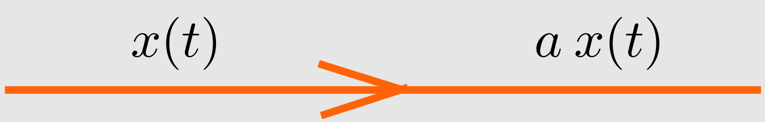
$$y[n] = \frac{1}{3}\{x[n] + x[n-1] + x[n-2]\}$$

Representación de sistemas mediante diagramas de bloques:

Sistemas continuos:

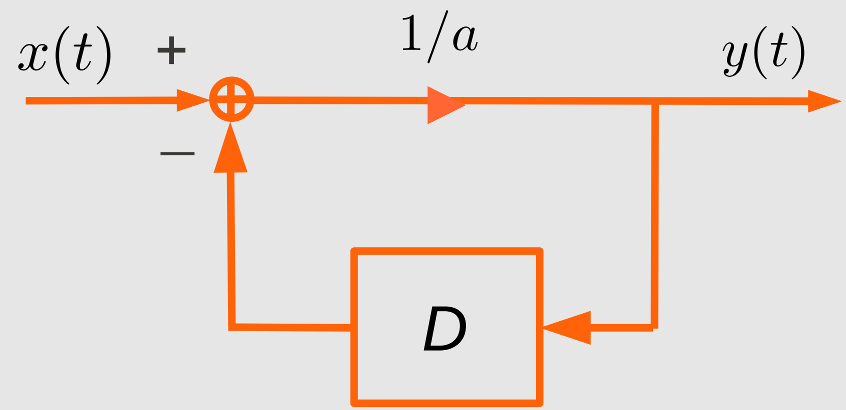
$$\frac{d^N y(t)}{dt^N} + \sum_{k=0}^{N-1} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$



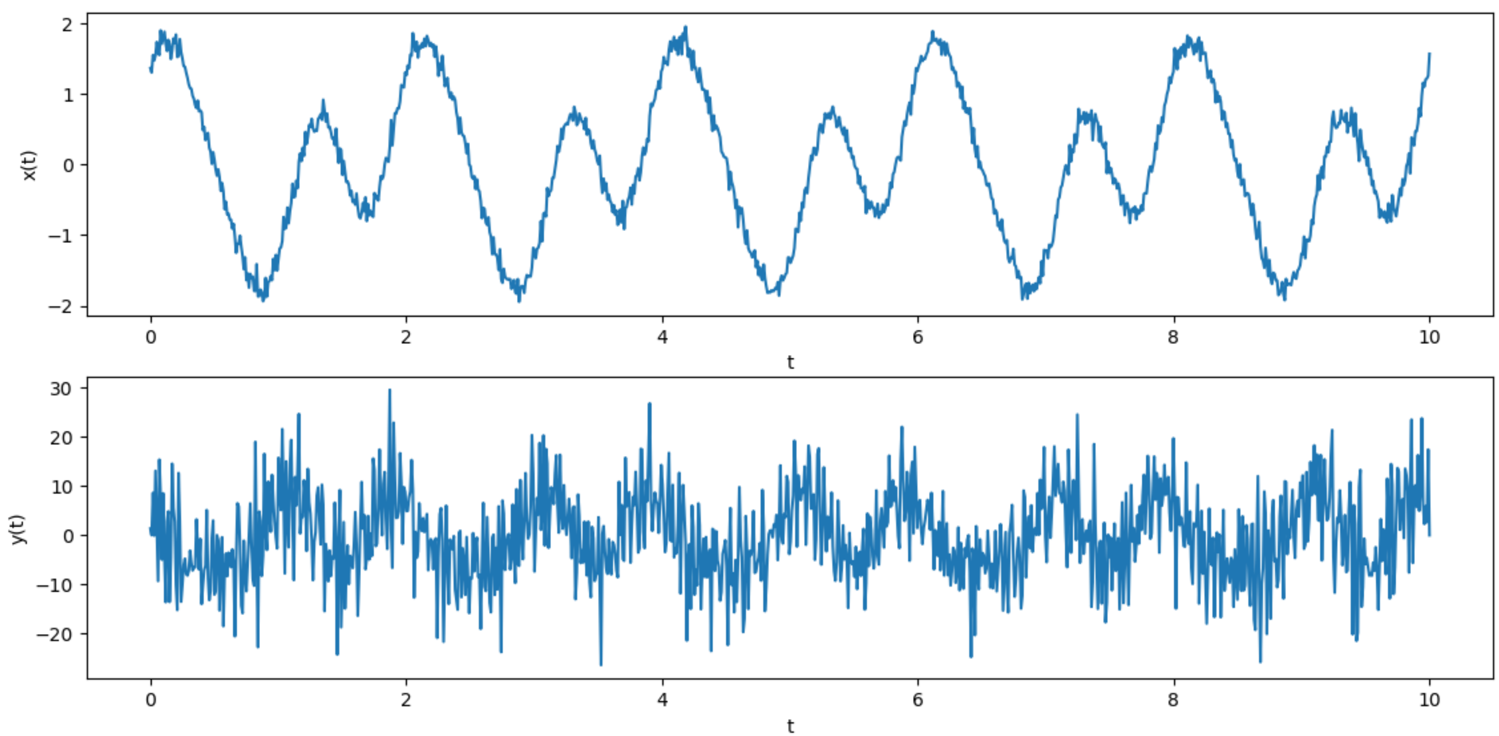


$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

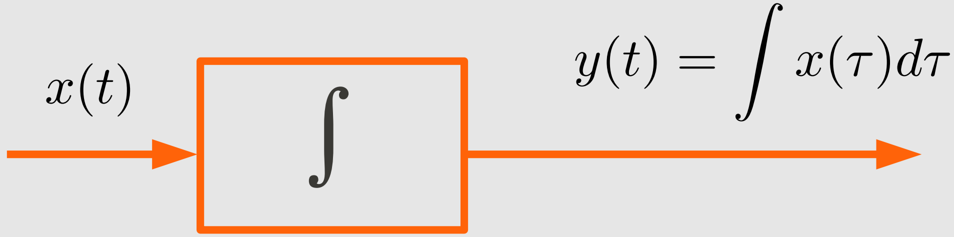
$$y(t) = \frac{1}{a}\left(x(t) - \frac{dy(t)}{dt}\right)$$



$x(t)$
D
$y(t) = \frac{dx(t)}{dt}$

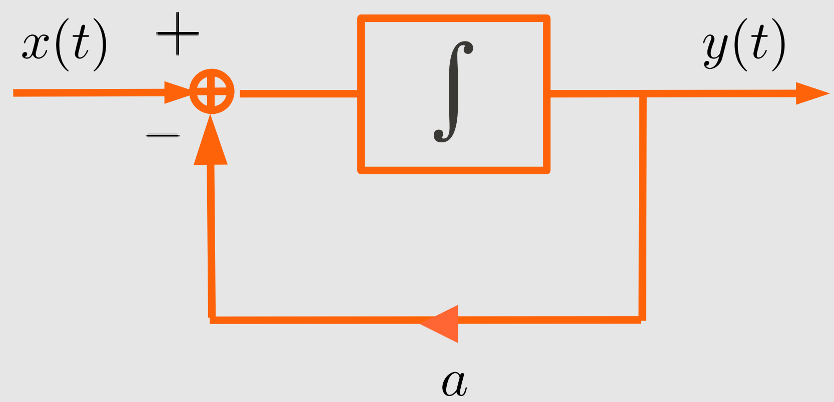


x(t)
2
1
0
−1
−2
0
2
4
6
8
10
t
y(t)
30
20
10
0
−10
−20
0
2
4
6
8
10
t



$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

$$\frac{dy(t)}{dt} = x(t) - ay(t)$$



$$y(t) = \int \left(x(\tau) - ay(\tau)\right) d\tau + c$$

$$y(t) = \int_0^t \left(x(\tau) - ay(\tau)\right) d\tau + y(0)$$